

# Vortex contribution to equilibrium currents

Zeev Vager

*Department of Particle Physics,  
Weizmann Institute of Science, 76100 Rehovot, Israel*  
(Dated: July 14, 2008)

For thin conducting rings, persistent currents were predicted to depend periodically on the external magnetic field. It is shown here that in general, whenever a periodic component of the current is found, there must be a large aperiodic component forcing aperiodic contribution to the equilibrium magnetic moment as a function of the external field. It is also shown that contrary to previous publications, the conduction electrons partition function at low temperatures is always aperiodic as a function of the external magnetic field. This finding is relevant to experimental observations of magnetism of conducting rings as well as general magnetization phenomena.

## A. Introduction

The general subject of persistent current in conducting rings is geared around the notion of its being periodic with the magnetic flux penetrating the rings. Many experimental measurements in normally conducting rings[1] confirm periodic components with a fundamental flux period of  $hc/e$  and its harmonics. However, the experimental currents are significantly higher than the theoretical predictions[2]. In the theoretical publications, the notion of periodicity is considered to be well known, as if given by gauge invariance. The purpose of this letter is to disagree with this notion and to show that a major aperiodic component has been neglected. In superconducting magnets this is evident since the persistent current is simply proportional to the penetrating quantized flux. As will be shown, each time persistent current exists in a ring, a large aperiodic component must persist too.

The notion of magnetic flux periodicity of the partition function of electrons within a superconductor ring, disregarding surface currents, was coined by Byers and Yang[3]. Clearly, the magnetic field of superconducting magnets does not show this periodicity because it is solely due to surface currents. This is not in contradiction with Byers and Yang which excluded electrons in the penetration depth region of the superconductor. This allowed Byers and Yang to use a restricted region electronic Hamiltonian where the magnetic field is strictly set to zero. Therefore, this restricted Hamiltonian cannot be used for prediction of surface currents.

However this restricted Hamiltonian was used in the literature for treating magnetization of rings. As a result of using the restricted Hamiltonian, one obtains a misleading notion of equilibrium current which depends periodically on the penetrating magnetic flux. The present article indicates some of those previous publications on equilibrium magnetization using such incorrect Hamiltonian which resulted periodic behavior. The leading article is by Bloch[4], treating a superconducting ring which contains a single barrier. Other articles followed this periodic notion claiming similar results for normal rings[2, 5]. As will be shown, an overlooked electromagnetic energy in the Hamiltonian demands that whenever

equilibrium persistent current is measured in rings the dependence on external magnetic field is always aperiodic. That does not exclude a periodic component, as measured in numerous occasions. The measurement of the periodic component of persistent current is facilitated by observation of harmonics of the natural fundamental component and therefore excludes the aperiodic component. This is how the misleading notion of periodicity of persistent current got its experimental fictitious support.

The correct approximation to the non-relativistic Hamiltonian and its implications to magnetization is also highly relevant to the understanding of experimental observations of magnetization that were left unexplained so far. In particular, the mechanism of giant magnetization of thin organic layers has been disputed[6] by the use of the restricted Hamiltonian. Therefore, revised considerations are needed.

## B. Preliminary remarks

This article is about magnetization of conductors under special conditions of equilibrium. At low enough temperatures, a phenomenological difference between a normal conductor and a superconductor is that the fraction of non-viscous conduction electrons in a normal conductor tends to zero for infinite length-dimensions while this fraction is independent of length in a superconductor. It will be shown that the mere existence of non-zero fraction of non-viscous electrons in a finite normal ring is enough for equilibrium aperiodic magnetization.

In particular, consider a conductor ring in a uniform magnetic field along the ring axis. Assume that there is a temperature where a non-zero fraction of the conduction electrons flows without viscosity (persistent current). This assumption leads to aperiodic magnetization in both the classical limit and in quantum mechanics.

The discussion starts with superconducting rings where flux is quantized and persistent currents are simply proportional to the trapped flux. In type I superconductors there are no transitions between the quantized persistent currents for a finite range of temperatures above zero. This is a case where the persistent current is isolated even from the environmental radiation temper-

ature. In any finite ring, both type II superconductors and normal conductors at low enough temperature have a non-viscous fraction of conduction electrons, sometimes, ever so small but non-zero. Ignoring radiation, the superconducting ring quantization of the non-viscous fraction is valid. Including radiation, equilibrium is established and the quantized currents have statistical weights resulting a major current component which is linear with the external field. The remaining periodic current component is generally smaller. Detailed discussion is given below.

### C. Superconducting rings

In 1961, a celebrated experiment by Deaver and Fairbank[7] had shown that superconducting hollow cylinders sustain quantized magnetic flux  $\phi_n$  which are integer,  $n$ , multiples of  $hc/2e$ . From that time it is accepted that superconducting rings carry currents which are proportional to the trapped quantized flux. If such rings are held in place (in relation to environment), the far magnetic-dipole field is quantized too. The quantization is known to hold within some limited range of temperatures and external magnetic fields. The flux dependent energy of a superconducting ring within a uniform external magnetic field  $B$  is given by

$$E_n(B) = \frac{1}{2\mathcal{L}}\phi_n^2 - \frac{1}{\mathcal{L}}AB\phi_n \equiv \frac{1}{2\mathcal{L}}\phi_n^2 - \frac{1}{\mathcal{L}}\Phi\phi_n \quad (1)$$

where  $\mathcal{L}$  is the inductance of the superconducting ring and  $A$  is an effective area[8] related to the ring. When the magnetic dipole of a superconducting ring is pointing along the external magnetic field,  $A$  is maximal and  $\Phi$  is the external flux through that area without the superconducting ring.

Plugging in the flux quantization, the energy expression is

$$E_n(B) = \frac{\hbar^2}{2\mathcal{I}}(n - \eta)^2 - \frac{\hbar^2}{2\mathcal{I}}\eta^2 \quad (2)$$

where

$$\mathcal{I} = 4\pi^2\alpha^2 m_0 a_0 \mathcal{L} = 4\pi^2 m_0 r_0 \mathcal{L} \quad \text{and} \quad \eta = \frac{\Phi}{hc/2e} \quad (3)$$

( $\alpha$  - fine structure constant,  $m_0$  - electron mass,  $a_0$  - Bohr radius and  $r_0 = e^2/m_0 c^2$ ). These are shifted rotational states with large rotational energies. Since the moment of inertia of a single electron on the cylinder is  $\mathcal{I}_0 = m_0(\mathcal{L}/2\pi)^2$  then

$$\mathcal{I} = \mathcal{I}_0/K \quad (4)$$

where

$$K = \frac{1}{(4\pi)^4} \frac{\mathcal{L}}{r_0}. \quad (5)$$

Even for atomic scale and especially for macroscopic or mesoscopic rings,  $K$  is a huge number. Conversely, the effective moment of inertia is by far smaller than  $\mathcal{I}_0$ . Most importantly, the phenomenon is independent of the non-viscous fraction of conduction electrons as long as it is non-zero.

The  $e^{-2}$  energy cofactor in (2) forces the need of non-perturbative solutions of the fields. Therefore, use of electromagnetic potentials within hamiltonian formulations should be treated with caution[9]. This is elaborated next. Lastly, it should be clear that the last term  $-\hbar^2\eta^2/2\mathcal{I} = -A^2B^2/2\mathcal{L}$  is of classical origin and signify the reduction of the overall magnetic energy for any quantum state.

### D. Formal note on non-relativistic hamiltonian

A non-relativistic approximation to the classical hamiltonian for charged particles in an electromagnetic field is

$$\mathcal{H} = \sum_n \frac{1}{2m_n} (\vec{p}_n - \frac{e_n}{c} \vec{A})^2 + U + \frac{1}{8\pi} \int (B^2 + E^2) d\tau \quad (6)$$

where  $(U, \vec{A})$  is the relativistic vector potential which defines the fields  $\vec{B}, \vec{E}$ . It is understood that this vector is defined up to a gauge transformation. Such transformations do not change observations or fields. The total electromagnetic energy is a function of charge and current density. Therefore, the last term turns to an operator in the quantum Hamiltonian.

Certainly, the eigenvalues in (2) are eigenvalues of such a hamiltonian. Adding to the last term in (2), the positive and much larger constant  $\int B_0^2 d\tau$  (here  $B_0$  is the external field), completes the integration on the energy of the electromagnetic field. As mentioned above, the eigenvalues (2) result from nonperturbative treatment of the equations of motion. Specifically, the electromagnetic field cannot be treated as a perturbation.

The last term in (2) cannot be derived from a hamiltonian of the form

$$\mathcal{H} = \sum_n \frac{1}{2m_n} (\vec{p}_n - \frac{e_n}{c} \vec{A})^2 + U \quad (7)$$

since the classical reduction of the magnetic energy (see above) is missing.

Whenever equilibrium current exists in a loop, there is a magnetic dipole far field and the associated flux lines can be traced going in and out the current boundary. The current and this flux are proportional to each other with a constant of proportionality which depends on the geometry of the loop. This is a classical relation which stems from Maxwell equations. The function of the dipole self energy together with its interaction with external field has a minimum at a non-zero dipole moment.

As will be explained below, topologically, this situation is consistent with an Abrikosov-Nielsen-Olesen vortex. The far dipole field defines an axis along which north

and south poles reside. As shown by Wu and Yang[11], for any vector potential reproducing the dipolar magnetic field there is, at least, a section on the dipolar axis where the vector potential is undefined. Different vector potentials which reproduce the dipolar magnetic field and are well defined on that section are undefined on, at least, another section along the axis. Since the current carrying charged particles do not pass through the axis connecting the poles, any of the vector potentials reproducing the electromagnetic fields can be used within the hamiltonian provided that it explicitly contains the electromagnetic energy operator. It is concluded that the quantum hamiltonian of circulating current in a loop must include the magnetic energy like (6).

### E. Aharonov-Bohm flux and conducting rings

There is a well known theorem that if a magnetic flux  $\phi$  penetrates through a conducting ring without passing through the conductor (Aharonov-Bohm (AB) flux), all equilibrium physical properties of the ring are periodic with  $\phi$ . The period is given by  $hc/e$ . Under such conditions, if there is an equilibrium current then it is periodic with  $\phi$ .

How does one understand the currents in superconducting rings where persistent currents are aperiodic by simply being proportional to the flux within the ring? The answer is that there is a thin inside layer where the magnetic field penetrates and interacts with the circulating current. This is topologically equivalent to a flux line penetrating a type II superconductor accompanied by a super-current starting at zero at the flux line and decaying exponentially after a small radial distance (compared with the macroscopic dimensions of the superconductor). Such phenomenon is known as Abrikosov-Nielsen-Olesen vortex line which is a soliton solution to the relativistic field equation. In a superconducting ring, the current in the thin penetration depth and the flux within the ring interact with each other to form a 'fat' Abrikosov-Nielsen-Olesen vortex 'line'. It is **NOT** an AB flux which does not penetrate the conductor by definition.

Conversely, if an 'infinite' thin coil carrying a flux  $\phi_{AB}$  is inserted to any loop, its positive self energy is proportional to  $\phi_{AB}^2$ . If the loop is superconducting and no other flux penetrates it then there is an additional quantum term to the total energy which is  $\hbar^2(n - \eta_{AB})^2/2\mathcal{I}$ . There is no reduction of the magnetic energy due to interaction of the dipole field in the external field.

In a ring conductor having non-viscous flow, not necessarily superconducting, flux penetrates the conductor as well as the hole in the ring. Eigenenergies like (2) are a sum of two terms. The first term is proportional to  $(n - \eta)^2$  and behaves as if it is an AB flux. The second term is of classical origin and much larger, except near zero flux. This is further explained below.

### F. Equilibrium

When leakage of flux is possible (such as in type II superconductors), equilibrium can be achieved with the thermal radiation. The large rotational energy factor in (2) selects Boltzmann statistical factors such that the equilibrium is very near the one quantized flux which is nearest to  $\Phi$  (where  $|n - \eta|$  is minimal). Thus, at equilibrium, paramagnetic magnetization is reached. The partition function is **NOT** periodic with  $\eta$ . As will be discussed later, this aperiodic dependence on  $\Phi$  contributes a smooth behavior of the partition function. If the parabolic classical energy function[10]

$$\bar{E}(\eta) = -\frac{\hbar^2\eta^2}{2\mathcal{I}} = -\frac{A^2}{2\mathcal{L}}B^2 \quad (8)$$

is artificially subtracted from (2) then it becomes periodic with  $\eta$ . But this energy part of (2) is clearly paramagnetic. Its simple physical interpretation is that non-zero equilibrium persistent current creates a magnetic dipole which minimizes the self energy of the dipole together with the dipole interaction energy. It is interesting to compare it with deductions of Bloch[4] where the parabolic dependence is missing. As mentioned above, this is just one of many references where persistent current is considered as periodic with the external field.

### G. Normal conductor rings

The extension to normal conductors is realized by recognizing that once a non-zero fraction of the conduction electrons is non-viscous, the soliton energy dependence remains exactly as above. The only difference from type II superconductors is the rate of relaxation to equilibrium. The classical arguments are as follows.

Consider a normal ring in an external uniform magnetic field  $B$  which is in the direction of the ring axis. If a current  $I$  circulates in the ring then the current dependent energy of the system is

$$U = \frac{1}{2}\mathcal{L}I^2 - ABI \quad (9)$$

which is the classical analogue of (1). The main difference here is that the effective area  $A$  is significantly larger than the inside hole, yet smaller than the total area. An immediate consequence is that at thermal equilibrium (with radiation) there is a mean current

$$I_0 = AB/\mathcal{L}. \quad (10)$$

In terms of  $I_0$  the energy dependence on  $I$  is

$$U = \frac{1}{2}\mathcal{L}(I - I_0)^2 - \frac{1}{2}\mathcal{L}I_0^2 \quad (11)$$

which is negative at the vicinity of  $I_0$ . It is concluded that the ring is paramagnetic in nature. Moreover, the

magnetic moment at thermal equilibrium is given by

$$m = -\frac{\partial U}{\partial B}|_{I_0} = \frac{A^2}{\mathcal{L}}B. \quad (12)$$

This is the classical magnetic moment of the ring in an external field  $B$ , provided that there is a nonzero fraction of non-viscous conducting electrons. This could be measured by a magnetometer. The equilibrium energy of the ring is lowered by the existence of a magnetic field  $B$  by the negative energy

$$-mB = -\frac{A^2}{2\mathcal{L}}B^2 = -\frac{\Phi^2}{2\mathcal{L}}. \quad (13)$$

This energy is just the classical function in (8).

It should be noted that whatever quantum considerations dictate, the correspondence principle require that in the classical limit, the free energy dependence on  $I$  must be of form (11). This is indeed consistent with the behavior of superconducting rings, as in (2).

## H. Conclusion

If the interest is restricted to circulating currents in rings, an effective hamiltonian can be formulated which includes the soliton energy and the correct dependence on external magnetic field. The eigen-energies for a superconducting ring are given by (2). When the Meissner effect is not applicable but non-viscous currents exist, the only perturbation comes from thermal radiation. Therefore, Boltzmann factors can be define for each eigen-energy and the partition function is well defined.

Each eigen-energy is a sum of two terms

$E_n(B) = E_n^{(Q)}(B) + E^{(C)}(B)$  where

$$E_n^{(Q)}(B) = \frac{\hbar^2}{2\mathcal{I}}(n - \eta)^2 \quad \text{and} \quad (14)$$

$$E^{(C)}(B) = -\frac{A^2}{2\mathcal{L}}B^2. \quad (15)$$

Thus, the partition function,  $Z$ , has a common factor:  $\exp[-\beta E^{(C)}(B)]$ . Two terms contribute to the magnetization when the derivative of  $Z$  is taken with respect to  $B$ : a stable classical term (12) and a term where a statistically weighted sum of alternating sign magnetization[12]. The second term contribution[5] depends on details of the conductor and its theoretical predictions underestimate[2, 13] experimental results[1] of the periodic component. The first classical term is quite illusive and, so far, found only indirectly on molecular monolayers[14]. Obviously, such magnetization appear in any current loop where losses are compensated by power supplies.

The 'classical' magnetic energy which must be added to the eigenenergies of the hamiltonian (7) is the additional 'classical' energy of the separable hamiltonian (6). Similarly, the current amplitude is a superposition of current amplitudes of those two separable parts.

This article emphasizes the importance of the previously ignored classical magnetic energy to the free energy of conduction electrons. The phenomenon of aperiodic magnetization should appear in any of the persistent current experiments and many other magnetization phenomena.

Illuminating discussions with Prof. A. Schwimmer as well as a lesson in classical field theory are acknowledged.

- 
- [1] For example,  
L. P. Levy et al Phys. Rev. Lett. **64**, 2074 (1990).  
V. Chandrasekhar et al, Phys. Rev. Lett. **67**, 3578 (1991).  
D. Mailly et al Phys. Rev. Lett. **70**, 2020 (1993).
- [2] For example,  
H.F. Cheung et al, Phys. Rev. Lett. **62**, 587 (1989).  
A. Schmid, Phys. Rev. Lett. **66**, 80 (1991).  
F. von Oppen and E. Riedel, Phys. Rev. Lett. **66**, 84 (1991).  
B. L. Altshuler, Y. Gefen and Y. Imry, Phys. Rev. Lett. **66**, 88 (1991).
- [3] N. Byers and C. N. Yang, Phys. Rev. Lett. **7**, 46 (1961).
- [4] F. Bloch Phys. Rev. Lett, **21**, 1241 (1968).
- [5] M. Büttiker, Y. Imry and R. Landauer, Physics Letters **96A**, 365 (1983).
- [6] A. Hernando and M. A. Garcia, Phys. Rev. Lett. **96**, 029703 (2006)
- [7] Deaver and Fairbank, Phys. Rev. Lett. **7**, 43 (1961).
- [8] An effective area can be chosen so that the energy is given by (1). The electromagnetic units are chosen such that  $\mathcal{L}$  has length units.
- [9] P. A. M. Dirac, Proceedings of the Royal Society of London **A133**, 60, (1931).
- [10] For the total energy, this negative energy is added to the much larger positive homogeneous magnetic energy of the external field in order to get the full energy of the system. The condition  $A^2/\mathcal{L} \ll V$  is compatible with the 'homogeneous field' demand, where  $V$  is the volume of the field.
- [11] T. T. Wu, C. N. Yang, Phys. Rev. D **12**, 3845 (1975).
- [12] The order of the energy levels is given by the ordered  $(n - \eta)^2$  therefore the derivative with respect to  $B$  has alternating signs for ordered energy levels.
- [13] It is this author opinion that the article arXiv:0709.0589 from 2007 resolves the experiment-theory discrepancy.
- [14] Ron Naaman and Zeev Vager, to be published.